Primal-dual methods for solving infinite-dimensional games

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Outline

1. Convex-concave problem
2. Strongly convex-concave problem

Motivation

- We have two controlled objects. First player chooses control for the first object as a function of time. Second player controls the second object.

- We have some performance index depending on controls (energy or fuel consumption) and on coordinates of objects at final time moment (how close they are to each other).
- Players have opposite goals: first – minimize performance index, second – miximize.

Problem statement

We have two objects with motion given by equations

$$\frac{dx(t)}{dt} = A_x(t)x(t) + B(t)u(t), \; x(t) \in \mathbb{R}^n, \; u(t) \in P \subset \mathbb{R}^p$$
$$\frac{dy(t)}{dt} = A_y(t)y(t) + C(t)v(t), \; y(t) \in \mathbb{R}^m, \; v(t) \in Q \subset \mathbb{R}^q$$
$$(x(0), y(0)) = (x_0, y_0), \; t \in [0, \theta].$$

Performance index:

$$F(u, v) + \Phi(x, y) = \int_0^\theta \widetilde{F}(\tau, u(\tau), v(\tau))d\tau + \Phi(x(\theta), y(\theta)). \qquad (1)$$

Our goal: $u(t) \in L_2([0, \theta], P), \quad v(t) \in L_2([0, \theta], Q)$ s.t. (u, v) is a saddle point of (1).

Historical notes

Differential games

- Isaacs R., 1965, Hamilton-Jacobi-Bellman-Isaacs equation
- Pontryagin L.S., 1967, Alternating integral method
- Krasovskii N.N., Subbotin A.I., 1988, Stable bridge method
- Ongoing research by many people, see review Kumkov, S.S., Le Ménec, S., Patsko, V.S., 2017

Optimization

- Mirror Descent, Nemirovsky A.S., Yudin D.B., 1983; Beck A., Teboulle M., 2003
- Dual averaging, Nesterov Y., 2009
- Extragradient method, Korpelevich G.M., 1976
- Mirror-Prox, Nemirovski A., 2004
- Dual extrapolation, Nesterov, Y., 2007
- Ongoing research by many people...

Assumptions

Introduce operators $\mathcal{B} : L_2([0,\theta],P) \to \mathbb{R}^n, \quad \mathcal{C} : L_2([0,\theta],Q) \to \mathbb{R}^m$:

$$x(\theta) = V_x(\theta,0)x_0 + \int_0^\theta V_x(\theta,\tau)B(\tau)u(\tau)d\tau = x_0 + \mathcal{B}u,$$

$$y(\theta) = V_y(\theta,0)y_0 + \int_0^\theta V_y(\theta,\tau)C(\tau)v(\tau)d\tau = y_0 + \mathcal{C}v.$$

Assumptions

- There exist a saddle-point.
- The sets P, Q are closed, convex and bounded.
- In the performance index $F(u,v) + \Phi(x,y)$
  - $F(\cdot, v)$ is convex for any fixed v,
  - $F(u, \cdot)$ is concave for any fixed u,
  - $\Phi(\cdot, y)$ is convex for any fixed y,
  - $\Phi(x, \cdot)$ is concave for any fixed x.
- $F(u,v)$ is u.s.c. in v and l.s.c. in u, and $\Phi(x,y)$ is continuous.
- Operators $\mathcal{B}, \mathcal{C}$ are bounded.

Transform problem

We solve the problem

$$\min_{u\in\mathcal{U}}\left[\max_{v\in\mathcal{V}}\{F(u,v)+\Phi(x,y) : y=y_0+\mathcal{C}v\} : x=x_0+\mathcal{B}u\right].$$

Since $\mathcal{B},\mathcal{C}$ are bounded, $x(\theta), y(\theta)$ are bounded and we can assume that $x(\theta)\in X, y(\theta)\in Y$, where X, Y are closed, convex and bounded.

Introducing Lagrange multipliers λ, μ, we write an adjoint problem

$$\begin{aligned}&\min_{\lambda}\max_{\mu}\left\{\min_{u\in\mathcal{U}}\max_{v\in\mathcal{V}}[F(u,v)-\langle\mu,\mathcal{B}u\rangle+\langle\lambda,\mathcal{C}v\rangle]+\right.\\&\left.+\min_{x\in X}\max_{y\in Y}[\Phi(x,y)+\langle\mu,x\rangle-\langle\lambda,y\rangle]-\langle\mu,x_0\rangle+\langle\lambda,y_0\rangle\right\}\\&=:\min_{\lambda}\max_{\mu}\psi(\lambda,\mu)\end{aligned}$$

Lemma

Initial problem is equivalent to the adjoint problem.

Reformulation as Variational Inequality

Let (u^*, v^*, x^*, y^*) be saddle-point in the definition of $\psi(\lambda, \mu)$ with fixed λ, μ. Then

- $\psi(\cdot, \mu)$ is convex in λ for all μ and has subgradient $\psi'_\lambda(\lambda, \mu) = \mathcal{C}v^* + y_0 - y^* \in \partial_\lambda \psi(\lambda, \mu)$ which is bounded.
- $\psi(\lambda, \cdot)$ is concave in μ for all λ and has supergradient $\psi'_\mu(\lambda, \mu) = x^* - \mathcal{B}u^* - x_0 \in \partial_\mu \psi(\lambda, \mu)$ which is bounded.

Problem reformulation

- Saddle-point: $\psi(\lambda^*, \mu) \leq \psi(\lambda^*, \mu^*) \leq \psi(\lambda, \mu^*) \quad \forall \lambda, \mu$
- Concavity in μ: $\psi(\lambda, \mu^*) \leq \psi(\lambda, \mu) + \langle \psi'_\mu(\lambda, \mu), \mu^* - \mu \rangle \quad \forall \lambda, \mu$
- Convexity in λ: $\psi(\lambda^*, \mu) \geq \psi(\lambda, \mu) + \langle \psi'_\lambda(\lambda, \mu), \lambda^* - \lambda \rangle \quad \forall \lambda, \mu$
- $\langle \psi'_\lambda(\lambda, \mu), \lambda - \lambda^* \rangle + \langle -\psi'_\mu(\lambda, \mu), \mu - \mu^* \rangle \geq 0 \quad \forall \lambda, \mu.$

Denote $z = (\lambda, \mu)$, $g(z) = (\psi'_\lambda(\lambda, \mu), -\psi'_\mu(\lambda, \mu))$. Our goal is to find a weak solution z^* of the variational inequality

$$\langle g(z), z - z^* \rangle \geq 0 \quad \forall z \in S (\equiv \mathbb{R}^n \times \mathbb{R}^m).$$

General finite-dimensional VI

$$\text{Find} \quad z^* \quad \text{s.t.} \quad \langle g(z), z - z^* \rangle \geq 0 \quad \forall z \in S,$$

- $S \subseteq \mathbb{R}^N$ – convex closed set,
- $g(z)$ – bounded and monotone operator, i.e. $\langle g(z_1) - g(z_2), z_1 - z_2 \rangle \geq 0$.

Some auxiliary objects

- Choose norm $\|\cdot\|$ in $\mathbb{R}^N$ and σ-strongly convex prox-function $d(z)$, i.e., for any $z_1, z_2 \in S$, $d(z_2) - d(x) - \langle \nabla d(z_1), z_2 - z_1 \rangle \geq \frac{\sigma}{2}\|z_1 - z_2\|^2$.
  - Euclidean setup: $\|\cdot\| = \|\cdot\|_2$, $d(x) = \frac{1}{2}\|x\|_2^2$.
  - Simplex setup: $S = \{z \in \mathbb{R}^n_+, \langle e, z \rangle = 1\}$, $\|\cdot\| = \|\cdot\|_1$, $d(x) = \sum_{i=1}^n x_i \ln x_i$.
- $D : d(z^*) \leq D$, $\mathcal{F}_D = \{z \in S : d(z) \leq D\}$
- Given sequences $\lambda_i \geq 0$, $z_i \in S$, $g_i \in \mathbb{R}^N$, $i = 0, \dots, k$, define

$$\delta_k(D) = \max_z \left\{ \sum_{i=0}^k \lambda_i \langle g_i, z_i - z \rangle : \quad z \in \mathcal{F}_D \right\},$$

- sequence $\hat{\beta}_i$: $\quad \hat{\beta}_0 = \hat{\beta}_1 = 1, \quad \hat{\beta}_{i+1} = \hat{\beta}_i + \frac{1}{\hat{\beta}_i}$. NB: $\hat{\beta}_k \sim \sqrt{2k}$.

Simple Dual Averages Method [Nesterov, 2009]

Initialization $s_0 = 0, z_0, \gamma > 0$
Step $k \geq 0$

1. $g_k = g(z_k)$. $\quad s_{k+1} = s_k + g_k$.
2. $\beta_{k+1} = \gamma \hat{\beta}_{k+1}$. $\quad z_{k+1} = \arg\min_{z \in S} \{ \langle s_{k+1}, z \rangle + \beta_{k+1} d(z) \}$.

Theorem (Nesterov, 2009)

Assume that $\|g_k\|_* \leq L, k \geq 0$.

1. $\delta_k(D) \leq \hat{\beta}_{k+1} \left(\gamma D + \frac{L^2}{2\sigma\gamma} \right)$.
2. *If a solution* z^* *exists, then* $\|z_k - z^*\|^2 \leq \frac{2}{\sigma} d(z^*) + \frac{L^2}{\sigma^2 \gamma^2}$.
3. *If exists* $r > 0 : \mathfrak{B}_r(z^*) \subseteq \mathcal{F}_D$, *where* $\mathfrak{B}_r(z_0) = \{ z : \|z - z_0\| \leq r \}$, *then*

$$\frac{1}{k+1} \left\| \sum_{i=0}^{k} g_i \right\|_* \leq \frac{\hat{\beta}_{k+1}}{r(k+1)} \left(\gamma D + \frac{L^2}{2\sigma\gamma} \right).$$

Back to our problem

$$\text{Find} \quad z^* \quad \text{s.t.} \quad \langle g(z), z - z^* \rangle \geq 0 \quad \forall z \in \mathbb{R}^n \times \mathbb{R}^m,$$

where $z = (\lambda, \mu)$, $g(z) = (\psi'_\lambda(\lambda, \mu), -\psi'_\mu(\lambda, \mu))$ – bounded.

- Prox-functions $d_\lambda(\lambda)$ is strongly convex with convexity parameter σ_λ, $d_\mu(\mu)$ is strongly convex with convexity parameter σ_μ.
- $\|z\| = \sqrt{\kappa\sigma_\lambda \|\lambda\|^2 + (1-\kappa)\sigma_\mu \|\mu\|^2}$, $\kappa \in [0, 1]$.
- Define $d(z) = \kappa d_\lambda(\lambda) + (1-\kappa) d_\mu(\mu)$.
- Let (λ^*, μ^*) – be a saddle-point in the adjoint problem.
- Since z_k is bounded, we can choose D_λ s.t. $d_\lambda(\lambda_k) \leq D_\lambda \quad \forall k \geq 0$ and $\mathfrak{B}_{r/\sqrt{\kappa\sigma_\lambda}}(\lambda^*) \subseteq \{\lambda : d_\lambda(\lambda) \leq D_\lambda\}$ for some $r > 0$. Similarly we choose D_μ.
- Define $\mathcal{F}_D = \{z \in S : d(z) \leq D\}$, $D = \kappa D_\lambda + (1-\kappa) D_\mu$. Then $\mathfrak{B}_r(z^*) \subseteq \mathcal{F}_D$.

Equivalent problem

- Denote

$$\phi(u,x,v,y) = \min_{\lambda} \max_{\mu} \{F(u,v) + \Phi(x,y) + \langle \mu, x - x_0 - \mathcal{B}u \rangle + \langle \lambda, y_0 + \mathcal{C}v - y \rangle : d_\lambda(\lambda) \le D_\lambda, d_\mu(\mu) \le D_\mu\}.$$

- Then our problem is equivalent to $\min_{u \in \mathcal{U}, x \in X} \max_{v \in \mathcal{V}, y \in Y} \phi(u,x,v,y)$.
- Denote

$$\xi(u,x) = \max_{v \in \mathcal{V}, y \in Y} \phi(u,x,v,y),$$
$$\eta(v,y) = \min_{u \in \mathcal{U}, x \in X} \phi(u,x,v,y).$$

- Then $\xi(u,x) \ge \phi(u^*,x^*,v^*,y^*) \ge \eta(v,y) \quad \forall u \in \mathcal{U}, v \in \mathcal{V}, x \in X, y \in Y$.
- Duality gap $\xi(\hat{u},\hat{x}) - \eta(\hat{v},\hat{y})$ characterizes the quality of an approximate solution $(\hat{u},\hat{x},\hat{v},\hat{y})$.

Main result

Denote

$$\hat{u}_{k+1} = \frac{1}{k+1}\sum_{i=0}^{k} u_i, \quad \hat{v}_{k+1} = \frac{1}{k+1}\sum_{i=0}^{k} v_i,$$
$$\hat{x}_{k+1} = \frac{1}{k+1}\sum_{i=0}^{k} x_i, \quad \hat{y}_{k+1} = \frac{1}{k+1}\sum_{i=0}^{k} y_i,$$

(u_i, v_i, x_i, y_i) s.-p. defining $\psi(\lambda_i, \mu_i)$, where (λ_i, μ_i) is generated by SDA.

Theorem

$$\xi(\hat{u}_{k+1}, \hat{x}_{k+1}) - \eta(\hat{v}_{k+1}, \hat{y}_{k+1}) \le \frac{1}{k+1}\delta_k(D) = O\left(\frac{1}{\sqrt{k}}\right).$$

Theorem

$$\|x_0 + \mathcal{B}\hat{u}_{k+1} - \hat{x}_{k+1}\| \le \frac{\sqrt{\sigma_\mu}\delta_k(D)}{r(k+1)}, \quad \|y_0 + \mathcal{C}\hat{v}_{k+1} - \hat{y}_{k+1}\| \le \frac{\sqrt{\sigma_\lambda}\delta_k(D)}{r(k+1)}.$$

Numerical example

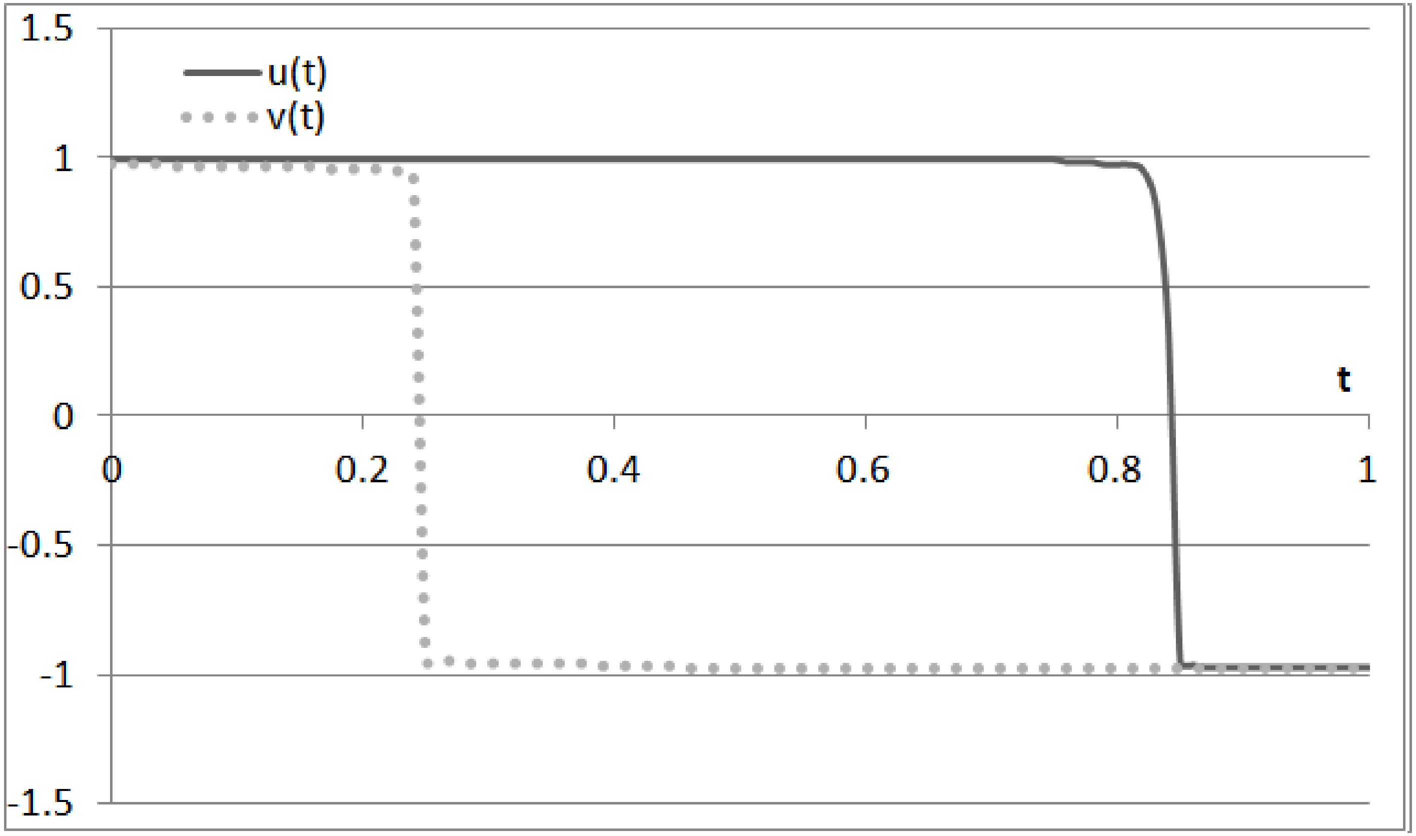
We have two objects with motion given by equations

$$\frac{dx(t)}{dt} = \begin{pmatrix} 1-t \\ 1 \end{pmatrix} u(t), \quad \frac{dy(t)}{dt} = \begin{pmatrix} 1-t \\ 1 \end{pmatrix} v(t), \quad u(t) \in P, v(t) \in Q$$

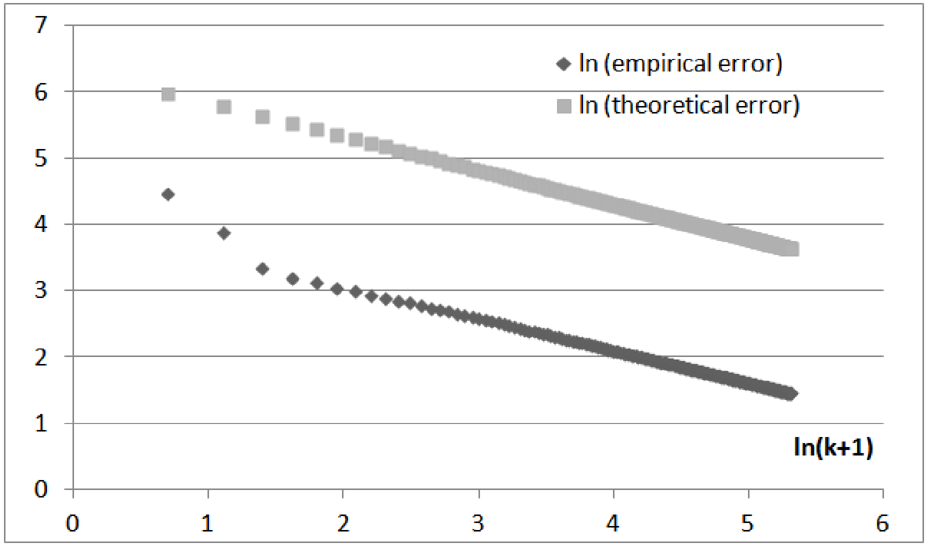
$t \in [0,1], \quad n = 2, \quad m = 2, P = Q = [-1,1].$

$$J(u,v) = \frac{1}{2}\|x(1) - y(1)\|^2 - \|y(1) - a\|^2.$$

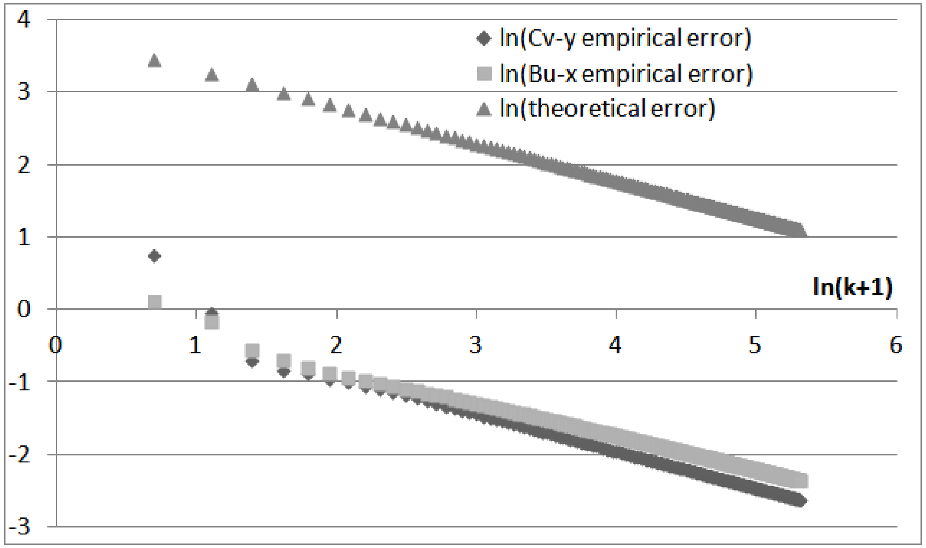
Controls



Error in funcional value



Error in equality consraints



Outline

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2. Strongly convex-concave problem

Additional assumptions

Assume additionally

- **Strong convexity**

  $F(u,v)$ is strongly convex in u with constant σ_{Fu} which doesn't depend on v and strongly concave in v with constant σ_{Fv} which doesn't depend on u,

  $\Phi(x,y)$ is strongly convex in x with constant $\sigma_{\Phi x}$ which doesn't depend on y and strongly concave in y with constant $\sigma_{\Phi y}$ which doesn't depend on x.

- **Lipschitz smoothness**

$$
\begin{aligned}
\|\nabla_u F(u,v_1) - \nabla_u F(u,v_2)\| &\le L_{uv}\,\|v_1 - v_2\|\,,\\
\|\nabla_v F(u_1,v) - \nabla_v F(u_2,v)\| &\le L_{vu}\,\|u_1 - u_2\|\,,\\
\|\nabla_x \Phi(x,y_1) - \nabla_x \Phi(x,y_2)\| &\le L_{xy}\,\|y_1 - y_2\|\,,\\
\|\nabla_y \Phi(x_1,y) - \nabla_y \Phi(x_2,y)\| &\le L_{yx}\,\|x_1 - x_2\|\,,\\
\|\nabla_x \Phi(x_1,y) - \nabla_x \Phi(x_2,y)\| &\le L_{xx}\,\|x_1 - x_2\|\,,\\
\|\nabla_y \Phi(x,y_1) - \nabla_y \Phi(x,y_2)\| &\le L_{yy}\,\|y_1 - y_2\|\,.
\end{aligned}
$$

Better properties of the adjoint problem

$$\min_{\lambda}\max_{\mu}\Bigg\{\min_{u\in\mathcal{U}}\max_{v\in\mathcal{V}}\left[F(u,v)-\langle\mu,\mathcal{B}u\rangle+\langle\lambda,\mathcal{C}v\rangle\right]+$$

$$+\min_{x}\max_{y}\left[\Phi(x,y)+\langle\mu,x\rangle-\langle\lambda,y\rangle\right]-\langle\mu,x_0\rangle+\langle\lambda,y_0\rangle\Bigg\}=\min_{\lambda}\max_{\mu}\psi(\lambda,\mu).$$

Let (u^*,v^*,x^*,y^*) be saddle-point in the definition of $\psi(\lambda,\mu)$ with fixed λ,μ.

Lemma

- *$\psi(\cdot,\mu)$ is convex and smooth in λ for any μ, $\nabla_\lambda\psi(\lambda,\mu)=y_0+\mathcal{C}v^*-y^*$ and $\nabla_\lambda\psi(\lambda,\mu)$ is Lipschitz continuous.*
- *$\psi(\lambda,\cdot)$ is concave and smooth in μ for any λ, $\nabla_\mu\psi(\lambda,\mu)=x^*-x_0-\mathcal{B}u^*$ and $\nabla_\mu\psi(\lambda,\mu)$ is Lipschitz continuous.*

Reformulation as Variational Inequality

$$\text{Find} \quad z^* \quad \text{s.t.} \quad \langle g(z), z - z^* \rangle \geq 0 \quad \forall z \in S,$$

- $z = (\lambda, \mu)$
- $S = \mathbb{R}^n \times \mathbb{R}^m$ – convex closed set,
- $g(z) = (\nabla_\lambda \psi(\lambda, \mu), -\nabla_\mu \psi(\lambda, \mu))$ – monotone operator,
- $g(z)$ is Lipschitz continuous, i.e., $\|g(z_1) - g(z_2)\|_* \leq L\|z_1 - z_2\|$.

Dual extrapolation method [Nesterov, 2007]

- Bregman divergence $\omega(x,y) = d(y) - d(x) - \langle \nabla d(x), y - x \rangle$,
- $\bar{x} \in \tilde{S}$ - center of S, $D : \omega(\bar{x}, x^*) \leq D$, $\mathcal{F}_D = \{x \in S : \omega(\bar{x}, x) \leq D\}$
- $T_\beta(z,s) = \arg\max_{x \in S}\{\langle s, x - z\rangle - \beta\omega(z,x)\}$.

Assume that $g(x)$ is Lipschitz continuous on S with constant L.
Initialization: Choose $\bar{x} \in \tilde{S}$. Fix $\beta = \frac{L}{\sigma}$. Set $s_{-1} = 0$.
Iteration ($k \geq 0$):

1. Compute $x_k = T_\beta(\bar{x}, s_{k-1})$,
2. Compute $y_k = T_\beta(x_k, -g(x_k))$,
3. Set $s_k = s_{k-1} - g(y_k)$.

Theorem [Nesterov, 2007]

- $\delta_k(D) := \max_x \left\{ \sum_{i=0}^k \lambda_i \langle g(y_i), y_i - x\rangle : \quad x \in \mathcal{F}_D \right\} \leq \frac{LD}{\sigma}$.
- If exists $r > 0 : \mathfrak{B}_r(x^*) \subseteq \mathcal{F}_D$, then $\frac{1}{k+1} \left\| \sum_{i=0}^k g(y_i) \right\|_* \leq \frac{LD}{\sigma(k+1)}$.

Main result

Denote

$$\hat{u}_{k+1} = \frac{1}{k+1}\sum_{i=0}^{k} u_i, \quad \hat{v}_{k+1} = \frac{1}{k+1}\sum_{i=0}^{k} v_i,$$

$$\hat{x}_{k+1} = \frac{1}{k+1}\sum_{i=0}^{k} x_i, \quad \hat{y}_{k+1} = \frac{1}{k+1}\sum_{i=0}^{k} y_i$$

(u_i, v_i, x_i, y_i) saddle-point defining $\psi(\lambda_i, \mu_i)$, where the sequence (λ_i, μ_i) is generated by the described method.

Theorem 1

$$\xi(\hat{u}_{k+1}, \hat{x}_{k+1}) - \eta(\hat{v}_{k+1}, \hat{y}_{k+1}) \leq \frac{LD}{k+1}.$$

NB: $\xi(u, x) \geq \phi(u^*, x^*, v^*, y^*) \geq \eta(v, y)$.

Theorem 2

$$\|x_0 + \mathcal{B}\hat{u}_{k+1} - \hat{x}_{k+1}\| \leq \frac{LD\sqrt{\sigma_\mu}}{r(k+1)}, \quad \|y_0 + \mathcal{C}\hat{v}_{k+1} - \hat{y}_{k+1}\| \leq \frac{LD\sqrt{\sigma_\lambda}}{r(k+1)}.$$

Numerical example

We have two objects with motion given by equations

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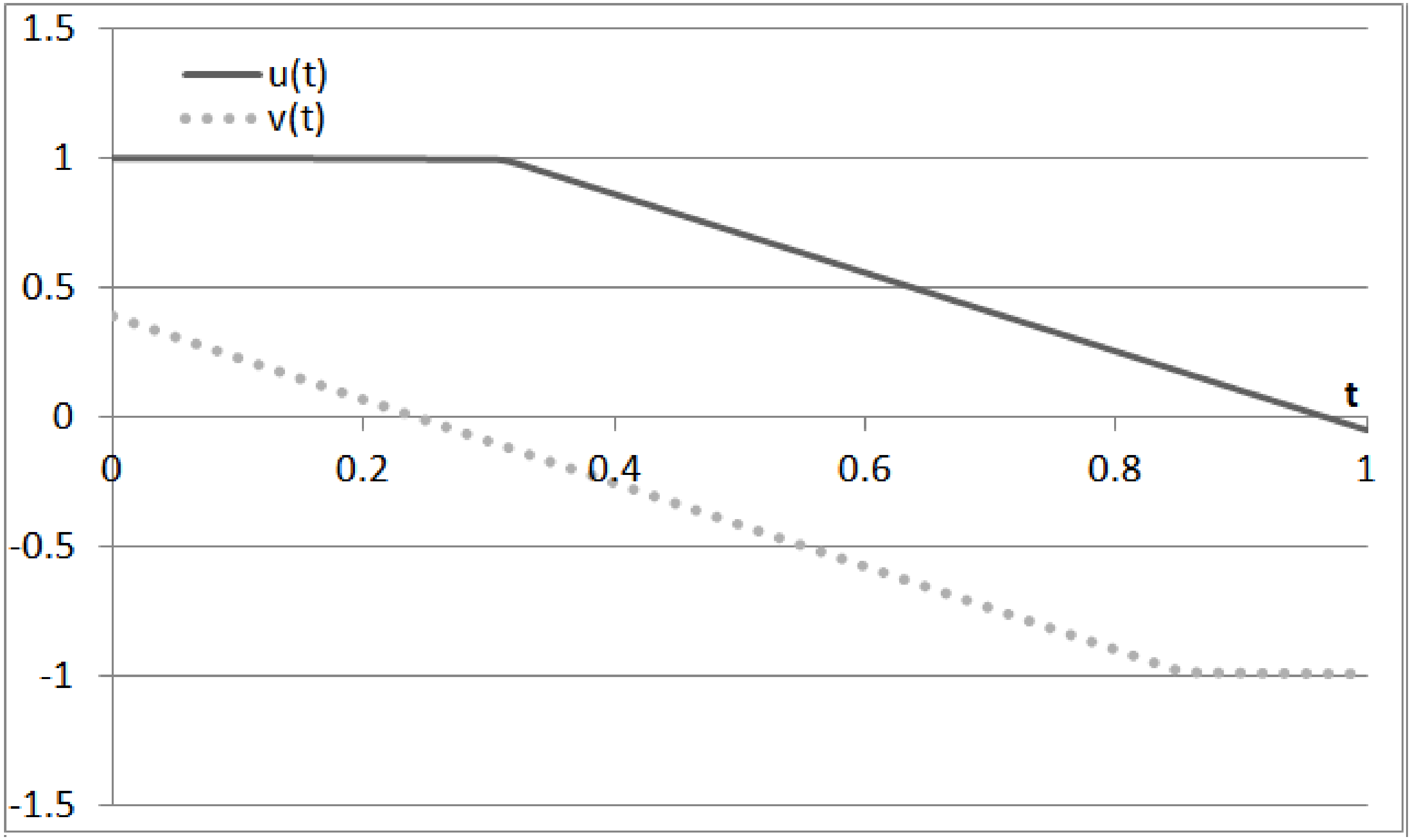
$t \in [0,1], \quad n = m = 2, P = Q = [-1,1].$

$$J(u,v) = \int_0^1 \left(\frac{(u(t))^2}{2} - \frac{(v(t))^2}{2} \right) dt + \frac{1}{2} \|x(1) - y(1)\|_2^2 - \|y(1) - a\|_2^2$$

Controls



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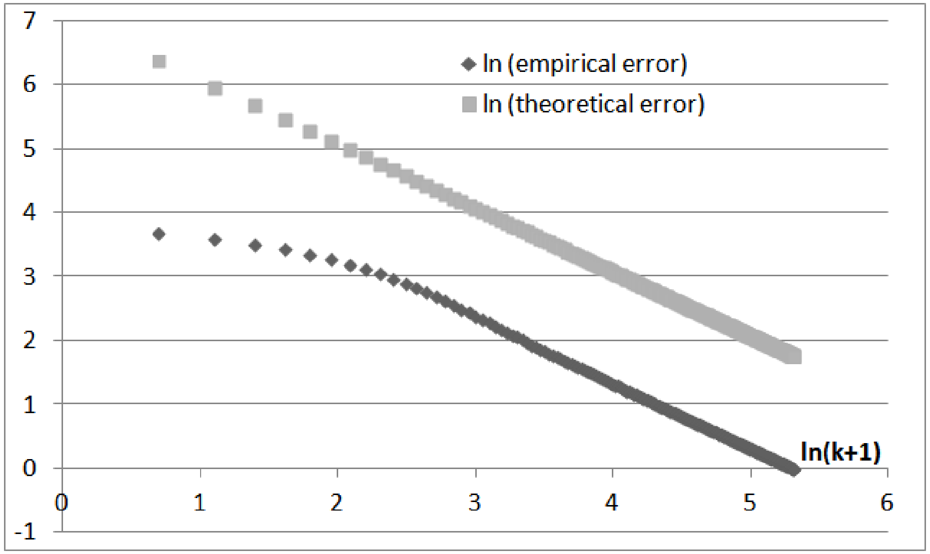

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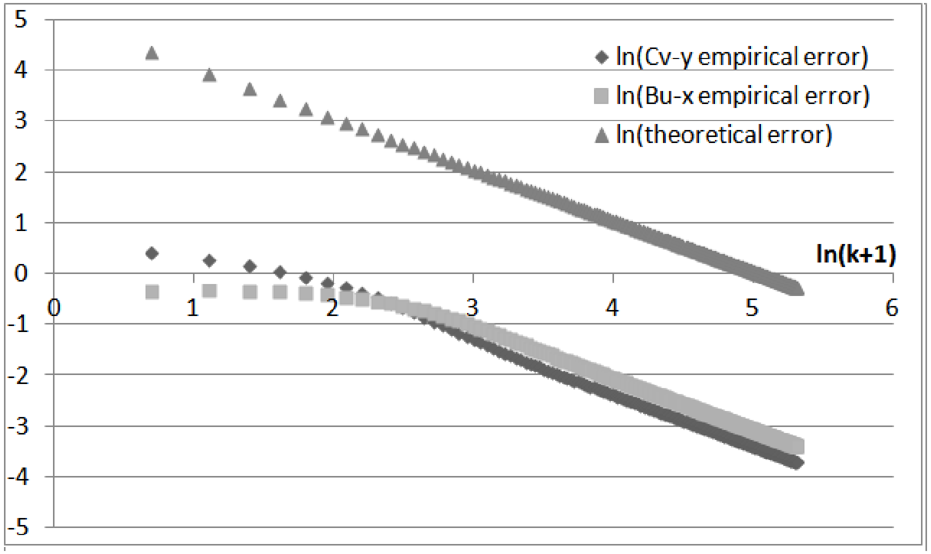

Conclusion

- We consider convex-concave and strongly convex-concave saddle-point optimal control problems (differential games).
- For each case, we propose an algorithm for approximating a saddle-point.
- We estimate the convergence rate of the proposed algorithms.
- Numerical experiments show that the practical performance is in consistency with the theoretical convergence rate estimates.

Thank you for your attention!