The Cone Condition and Nonsmoothness in Linear Generalized Nash Games

Oliver Stein

Institute of Operations Research
Karlsruhe Institute of Technology (KIT)

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This is joint work with

Nathan Sudermann-Merx, Advanced Business Analytics, BASF

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Survey

GNEPs in the general nonlinear setting

Given:

- N players $\nu \in \{1, \ldots, N\}$ with decision variables $x^\nu \in \mathbb{R}^{n_\nu}$,
- $x = (x^1, \ldots, x^N) = (x^\nu, x^{-\nu}) \in \mathbb{R}^n$ decision vector of all players,
- player ν's cost function $\theta_\nu(x^\nu, x^{-\nu})$,
- player ν's strategy space (= feasible set)

$$X_\nu(x^{-\nu}) \;=\; \{x^\nu \in \mathbb{R}^{n_\nu} |\; g^\nu(x^\nu, x^{-\nu}) \leq 0\},$$

Generalized Nash equilibrium problem

- player ν's optimization problem

$$Q_\nu(x^{-\nu}): \quad \min_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}) \quad \text{s.t.} \quad g^\nu(x^\nu, x^{-\nu}) \leq 0,$$

- player ν's optimal point set $S_\nu(x^{-\nu})$.

The generalized Nash equilibrium problem is

$$GNEP: \qquad \text{find some } \; x \in \bigcap_{\nu=1}^{N} \operatorname{gph} S_\nu \, .$$

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Literature review

- J. Nash, *Non-cooperative games,* Annals of Mathematics, Vol. 54 (1951), 286–295.
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Stylized example

Let two players share the constraints

$$g^1(x) \;=\; g^2(x) \;=\; \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -1 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} x_2 - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \;\leq\; 0$$

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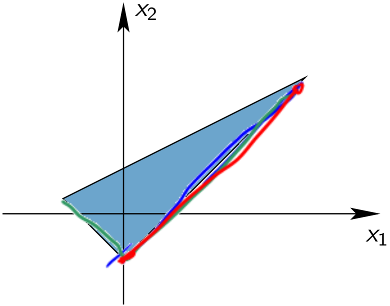

Restrictions on x

As $X_\nu(x^{-\nu})$ may be void for some choices of $x^{-\nu}$, it makes sense to restrict x to certain subsets of $\mathbb{R}^n$:

For

$$\Omega(x) \;=\; X_1(x^{-1}) \;\times\; \ldots \;\times\; X_N(x^{-N})$$

we call

$$M \;:=\; \operatorname{dom}\Omega \;=\; \{x \in \mathbb{R}^n |\ \Omega(x) \neq \emptyset\}$$

the common consistency set of all players.

Moreover

$$W \;:=\; \{x \in \mathbb{R}^n |\ g^\nu(x) \leq 0,\ \nu = 1, \ldots, N\} \;=\; \operatorname{fix}\Omega \;\subseteq\; \operatorname{dom}\Omega \;=\; M$$

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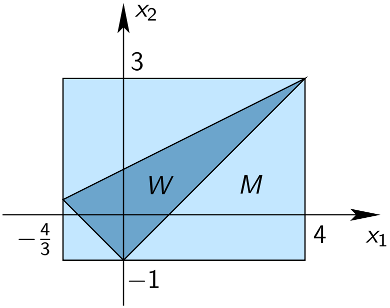

Optimal value functions and gap function

With the players' optimal value functions

$$\varphi_\nu(x^{-\nu}) \;:=\; \inf_{x^\nu \in X_\nu(x^{-\nu})} \theta_\nu(x^\nu, x^{-\nu}), \quad \nu = 1, \ldots, N,$$

the extended real-valued function

$$V(x) \;:=\; \sum_{\nu=1}^{N} \big(\theta_\nu(x^\nu, x^{-\nu}) - \varphi_\nu(x^{-\nu}) \big)$$

is called gap function of the GNEP.

We clearly have $V(x) > -\infty$ iff $x \in M$.

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$$\Rightarrow \quad \begin{aligned}\varphi_1(x_2) &= \begin{cases} -x_2 - 1 & , \ x_2 \in \operatorname{dom} X_1 = [-1, 3]\\ \infty & , \ \text{else}\end{cases}\\ \varphi_2(x_1) &= \begin{cases} |x_1| - 1 & , \ x_1 \in \operatorname{dom} X_2 = [-\frac{4}{3}, 4]\\ \infty & , \ \text{else,}\end{cases}\\ V(x) &= \begin{cases} 2 - x_1 - |x_1| + 2x_2 & , \ x \in M = [-\frac{4}{3}, 4] \times [-1, 3]\\ -\infty & , \ \text{else.}\end{cases}\end{aligned}$$

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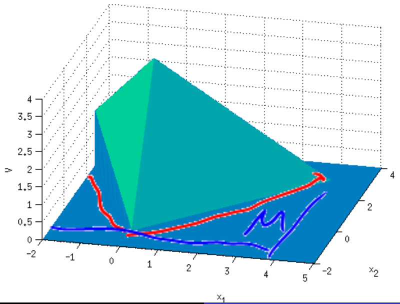

Reformulation as a single optimization problem

Assumption: For each $x \in M$ and each $\nu \in \{1, \dots, N\}$ the problem $Q^\nu(x^{-\nu})$ is solvable.

Then V is real-valued on $M \supseteq W$.

Theorem (v.Heusinger/Kanzow 2009, Dreves/Kanzow/St. 2012)

Generalized Nash equilbria are exactly the globally minimal points of the problem

$$\min\ V(x) \quad s.t. \quad x \in W$$

with minimal value zero.

What can be said about smoothness properties of V on W?

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Convexity and linearity assumptions

A GNEP is called

- player convex if for each ν and each $x^{-\nu}$ the function $\theta_\nu(\cdot, x^{-\nu})$ and all entries of $g^\nu(\cdot, x^{-\nu})$ are convex,
- jointly convex if for each ν the function θ_ν and all entries of g^ν are convex,
- player linear if for each ν and each $x^{-\nu}$ the functions $\theta_\nu(\cdot, x^{-\nu})$ and $g^\nu(\cdot, x^{-\nu})$ are linear,
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Continuity of V on W for player convex problems

For

$$\Omega(x) = X_1(x^{-1}) \times \ldots \times X_N(x^{-N})$$

we call

$$\operatorname{slater}\Omega = \{x \in \mathbb{R}^n |\ \Omega(x) \text{ satisfies the Slater condition}\}$$

the common Slater set of all players. Clearly, $\operatorname{slater}\Omega \subseteq M$.

Theorem (Dreves/Kanzow/St. 2012)

For each $x \in D = W \setminus \operatorname{slater}\Omega$ and each $\nu \in \{1, \ldots, N\}$ such that $X_\nu(x^{-\nu})$ violates the Slater condition, let $X_\nu(x^{-\nu}) = \{x^\nu\}$. Then V is continuous on W.

Stylized example

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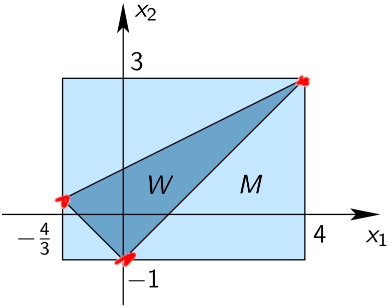

Directional differentiability of V for player convex problems

Proposition (Harms/Kanzow/St. 2015)

V is Hadamard directionally differentiable at each $x \in W \setminus D$ with

$$V'(x,d) \;=\; \sum_{\nu=1}^{N} \left[\langle \nabla\theta_\nu(x), d\rangle - \max_{\gamma^\nu \in \mathrm{KKT}_\nu(x)} \langle \nabla_x L_\nu(x, y^\nu(x), \gamma^\nu), d\rangle \right]$$

for all $d \in \mathbb{R}^n$.

Furthermore, in a certain sense the set of nondifferentiability points of V on W is small.

Hence, numerically one may try to apply a method for smooth optimization.

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Linear generalized Nash games

Next we consider completely linear GNEPs (LGNEPs):
For each $\nu \in \{1, \dots, N\}$ and $x \in \mathbb{R}^n$ let

$$Q_\nu(x^{-\nu}): \quad \min_{x^\nu \in \mathbb{R}^{n_\nu}} \langle a^\nu, x^\nu \rangle \quad \text{s.t.} \quad A^\nu x^\nu + B^\nu x^{-\nu} \leq c^\nu,$$

that is,

$$\theta_\nu(x^\nu, x^{-\nu}) = \langle a^\nu, x^\nu \rangle \quad (+f(x^{-\nu}))$$

and

$$X_\nu(x^{-\nu}) = \{x^\nu \in \mathbb{R}^{n_\nu} | \, A^\nu x^\nu \leq c^\nu - B^\nu x^{-\nu}\}.$$

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Applications

Applications: all linearly coupled LPs, like

- transportation problems with several forwarders (St./Sudermann-Merx 2018)
- linearly modelled profit centers
- routing in telecommunication or traffic networks
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Closest setting in the literature:
AGNEPs (Schiro/Pang/Shanbhag 2013)

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A global extension of V

For all $\nu = 1, \dots, N$ and $x^{-\nu} \in \operatorname{dom} X_\nu$ strong duality yields

$$\varphi_\nu(x^{-\nu}) \;=\; \max_{\sigma^\nu \in Z_\nu} \langle \sigma^\nu, B^\nu x^{-\nu} - c^\nu \rangle$$

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$Q(x^{-\nu})$ solvable $\;\Rightarrow\;$ $Z_\nu \neq \emptyset$.

$(\operatorname{dom} X_\nu)^c \neq \emptyset \;\Rightarrow\;$ Z_ν unbounded.

A global extension of V

Hence, with the non-empty vertex set

$$O_\nu \;:=\; \mathrm{vert}(Z_\nu)$$

we may define the global extension

$$\widehat{\varphi}_\nu(x^{-\nu}) \;:=\; \max_{\sigma^\nu \in O_\nu} \langle \sigma^\nu, B^\nu x^{-\nu} - c^\nu \rangle$$

of φ_ν on $\mathbb{R}^{n-n_\nu}$ as well as the global extension

$$\widehat{V}(x) \;:=\; \sum_{\nu=1}^{N} \langle a^\nu, x^\nu \rangle - \widehat{\varphi}_\nu(x^{-\nu})$$

of V on $\mathbb{R}^n$.

$\widehat{V}$ is polyhedral and concave on $\mathbb{R}^n$ and, hence, so is V on W.

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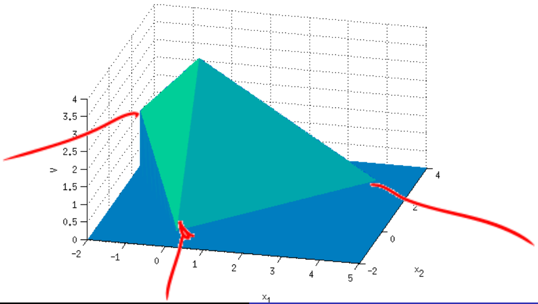

Localization and relation to the KKT set

For any $\nu = 1, \ldots, N$ and $x^{-\nu} \in \mathbb{R}^{n-n_\nu}$ let us define the active set

$$O_\nu(x^{-\nu}) \; := \; \{\sigma^\nu \in O_\nu | \; \langle \sigma^\nu, B^\nu x^{-\nu} - c^\nu \rangle = \widehat{\varphi}_\nu(x^{-\nu})\}.$$

Proposition (St./Sudermann-Merx 2016)

Around any $\bar{x} \in \mathbb{R}^n$ the function $\widehat{V}$ locally coincides with

$$\sum_{\nu=1}^{N} \langle a^\nu, x^\nu \rangle - \max_{\sigma^\nu \in O_\nu(\bar{x}^{-\nu})} \langle \sigma^\nu, B^\nu x^{-\nu} - c^\nu \rangle$$

Proposition (St./Sudermann-Merx 2016)

For any $\nu = 1, \ldots, N$ and $\bar{x}^{-\nu} \in \operatorname{dom} X_\nu$ we have

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Localization and relation to the KKT set

For any $\nu = 1, \ldots, N$ and $x^{-\nu} \in \mathbb{R}^{n-n_\nu}$ let us define the active set

$$O_\nu(x^{-\nu}) \;:=\; \{\sigma^\nu \in O_\nu |\; \langle \sigma^\nu, B^\nu x^{-\nu} - c^\nu \rangle = \widehat{\varphi}_\nu(x^{-\nu})\}.$$

Proposition (St./Sudermann-Merx 2016)

Around any $\bar{x} \in \mathbb{R}^n$ the function $\widehat{V}$ locally coincides with

$$\sum_{\nu=1}^{N} \langle a^\nu, x^\nu \rangle - \max_{\sigma^\nu \in O_\nu(\bar{x}^{-\nu})} \langle \sigma^\nu, B^\nu x^{-\nu} - c^\nu \rangle$$

Proposition (St./Sudermann-Merx 2016)

For any $\nu = 1, \ldots, N$ and $\bar{x}^{-\nu} \in \operatorname{dom} X_\nu$ we have

$$O_\nu(\bar{x}^{-\nu}) \;=\; \operatorname{vert}(\mathrm{KKT}(\bar{x}^{-\nu})).$$

The cone condition

Definition

For $\nu \in \{1, \ldots, N\}$ and $x^{-\nu} \in \mathbb{R}^{n-n_\nu}$ we say that the player cone condition (PCC) is valid in $x^{-\nu}$, if $O_\nu(x^{-\nu})$ contains at most one element. We say that the collective cone condition (CCC) holds in $x \in \mathbb{R}^n$, if PCC holds in $x^{-\nu}$ for all $\nu \in \{1, \ldots, N\}$.

Theorem (St./Sudermann-Merx 2016)

CCC in $\bar{x} \in \mathbb{R}^n$ implies the smoothness of $\widehat{V}$ in $\bar{x}$.

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Stylized example

$$\theta_1(x_1) = -x_1, \quad \theta_2(x_2) = x_2,$$

$$g^1(x) = g^2(x) = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -1 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} x_2 - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad Z_1 = \{\sigma^1 \in \mathbb{R}^3 : \ -1 - \frac{\sigma_1^1}{2} + \sigma_2^1 - \sigma_3^1 = 0, \ \sigma^1 \geq 0\}$$

$$O_1 = \{(0,1,0)^\intercal\}$$

$$|O_1(x_2)| = 1 \text{ for all } x_2 \in \mathbb{R}$$

$$Z_2 = \{\sigma^2 \in \mathbb{R}^3 : \ 1 + \sigma_1^2 - \sigma_2^2 - \sigma_3^2 = 0, \ \sigma^2 \geq 0\}$$

$$O_2 = \{(0,1,0)^\intercal, \ (0,0,1)^\intercal\}$$

$$|O_2(x_1)| = 1 \ \Leftrightarrow \ x_1 \neq 0.$$

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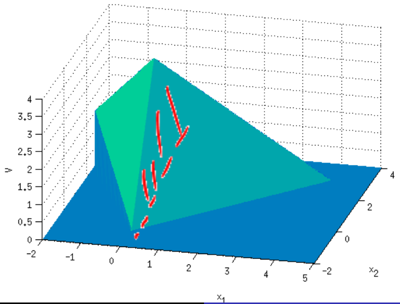

Characterization of smoothness by the cone condition

Theorem (St./Sudermann-Merx 2016)

For any $\nu \in \{1, \ldots, N\}$ and $J \subseteq \{1, \ldots, m_\nu\}$ with $|J| \leq n_\nu$ let the rows (A_j^ν, B_j^ν), $j \in J$, be linearly independent.
Then $\widehat{V}$ is smooth at $\bar{x} \in \mathbb{R}^n$ if and only if CCC holds at $\bar{x}$.

For $n_\nu = 1$, $\nu = 1, \ldots, N$, the above assumption can always be satisfied, otherwise it holds generically.

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Numerical consequences

Since the nonsmoothness structure of $\widehat{V}$ is well understood, and in particular, its Clarke subdifferential

$$\partial \widehat{V}(\bar{x}) = \sum_{\nu=1}^{N} \left\{ (a^\nu, -b^{-\nu}),\ b^{-\nu} \in \operatorname{conv}\{(B^\nu)^T \sigma^\nu,\ \sigma^\nu \in O_\nu(\bar{x}^{-\nu})\} \right\}$$

is easily evaluated, LGNEPs may be solved by nonsmooth optimization methods.

Numerical consequences

An exact penalty method as well as a projected subgradient method were successfully tested by Dreves/Sudermann-Merx 2016 as well as St./Sudermann-Merx 2018.

For example, the projected subgradient method solves LGNEPs with 100 players à 100 variables, that is, a nonsmooth nonconvex optimization problem with 10,000 variables, in about 2 minutes.

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Common KKT polyhedron

For $x \in \mathbb{R}^n$ we define the common KKT polyhedron

$$\Psi(x) \;:=\; \mathrm{KKT}_1(x^{-1}) \times \ldots \times \mathrm{KKT}_N(x^{-N}).$$

Lemma

- $x \in M \;\Leftrightarrow\; \Psi(x) \neq \emptyset.$
- $x \in \mathrm{slater}\,\Omega \;\Leftrightarrow\; \Psi(x)$ is bounded.
- If W possesses a Slater point, then $\mathrm{slater}\,\Omega = \mathrm{int}\, M$.

SMFC

Definition

For $\nu \in \{1, \ldots, N\}$ and $x^{-\nu} \in \operatorname{dom} X_\nu$, the player strict Mangasarian-Fromovitz condition (PSMFC) is valid in $x^{-\nu}$, iff $|\mathrm{KKT}_\nu(x^{-\nu})| \leq 1$. The collective strict Mangasarian-Fromovitz condition (CSMFC) holds in $x \in M$, if PSMFC holds in $x^{-\nu}$ for all $\nu \in \{1, \ldots, N\}$.

Lemma

CSMFC holds at $x \in M$ iff $|\Psi(x)| = 1$.

Proposition

Let W possess a Slater point. Then on $\operatorname{int} M$ the CCC and the CSMFC are equivalent.

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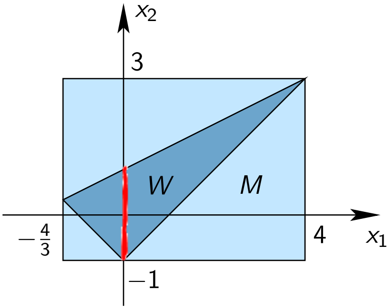
Common KKT polyhedron

Corollary

Let W possess a Slater point and let CCC be violated at $\bar{x} \in \mathrm{int}\, M$. Then for each $\nu \in \{1, \ldots, N\}$ with $|\,\mathrm{KKT}_\nu(\bar{x}^{-\nu})| > 1$ CCC also is violated at each $x \in \mathrm{int}\, M$ with $x^{-\nu} = \bar{x}^{-\nu}$.



Future research

- Effects of shared constraints.
- Extension to more general settings
  (e.g., jointly convex and separable GNEPs).
- Impact of the cone condition in different settings.
- Benders type cutting plane approaches.

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Reformulation as a single optimization problem

Let $\varphi_\nu(x^{-\nu}) = \inf_{x^\nu \in X_\nu(x^{-\nu})} \theta_\nu(x^\nu, x^{-\nu}), \quad \nu = 1, \ldots, N.$

Then:

$$
\begin{aligned}
x \in \bigcap_{\nu=1}^{N} \operatorname{gph} S_\nu \quad &\Leftrightarrow \quad x^\nu \in X_\nu(x^{-\nu}), \quad \theta_\nu(x^\nu, x^{-\nu}) = \varphi_\nu(x^{-\nu}) \quad \forall \nu \\
&\Leftrightarrow \quad x \in \Omega(x), \quad \sum_{\nu=1}^{N} \left(\theta_\nu(x^\nu, x^{-\nu}) - \varphi_\nu(x^{-\nu}) \right) = 0 \\
&\Leftrightarrow \quad x \in W, \quad V(x) = 0 \\
&\Leftrightarrow \quad x \text{ is minimal point of } V \text{ on } W \\
&\qquad\ \text{with minimal value } 0.
\end{aligned}
$$

Nikaido-Isoda function

We have

$$\begin{aligned} V(x) &= \sum_{\nu=1}^{N} \left(\theta_\nu(x^\nu, x^{-\nu}) - \varphi_\nu(x^{-\nu}) \right) \\ &= \sum_{\nu=1}^{N} \left(\theta_\nu(x^\nu, x^{-\nu}) - \min_{y^\nu \in X_\nu(x^{-\nu})} \theta_\nu(y^\nu, x^{-\nu}) \right) \\ &= \max_{y \in \Omega(x)} \sum_{\nu=1}^{N} \left(\theta_\nu(x^\nu, x^{-\nu}) - \theta_\nu(y^\nu, x^{-\nu}) \right). \end{aligned}$$

$$\psi(x,y) := \sum_{\nu=1}^{N} \left(\theta_\nu(x^\nu, x^{-\nu}) - \theta_\nu(y^\nu, x^{-\nu}) \right)$$

is called Nikaido-Isoda function.