Two modified Euler schemes

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SFB PDE

Gradient flows

Gradient flow

$$u'(t) + \mathrm{D}\phi(u(t)) = 0$$

$$\Downarrow \quad \not\Uparrow$$

Energy equality

$$\phi(u(t)) + \int_s^t \|u'\|^2 = \phi(u(s))$$

Euler scheme

Euler scheme

$$\frac{u_i - u_{i-1}}{\tau_i} + \mathrm{D}\phi(u_i) = 0$$

$$\not\Downarrow$$

Discrete energy equality

$$\phi(u_i) + \tau_i \left\| \frac{u_i - u_{i-1}}{\tau_i} \right\|^2 = \phi(u_{i-1})$$

Aim of this talk

Discuss two schemes recovering the discrete energy equality:

1. the Romero scheme
2. the De Giorgi scheme

1: The Romero scheme

Euler scheme:

$$\frac{u_i - u_{i-1}}{\tau_i} + \mathrm{D}\phi(u_i) = 0$$

The Romero scheme

$$\frac{u_i - u_{i-1}}{\tau_i} + \mathrm{D}\phi(u_i) + \Big(\phi(u_i) - \phi(u_{i-1}) - (\mathrm{D}\phi(u_i), u_i - u_{i-1})\Big)\frac{u_i - u_{i-1}}{\|u_i - u_{i-1}\|^2} = 0$$

Take the product with $(u_i - u_{i-1})$ and get the discrete energy equality

1: the Romero scheme

The Romero scheme

$$\frac{u_i - u_{i-1}}{\tau_i} + \mathrm{D}\phi(u_i) + \Big(\phi(u_i) - \phi(u_{i-1}) - (\mathrm{D}\phi(u_i), u_i - u_{i-1})\Big)\frac{u_i - u_{i-1}}{\|u_i - u_{i-1}\|^2} = 0$$

- Introduced in [Romero, 10] (no proofs)
- Asks for $u_i \neq u_{i-1}$ (or $\mathrm{D}\phi(u_i) = 0$)
- Stable scheme, because of the discrete energy identity

1: the Romero scheme

The Romero scheme

$$\frac{u_i - u_{i-1}}{\tau_i} + \mathrm{D}\phi(u_i) + \Big(\phi(u_i) - \phi(u_{i-1}) - (\mathrm{D}\phi(u_i), u_i - u_{i-1})\Big)\frac{u_i - u_{i-1}}{\|u_i - u_{i-1}\|^2} = 0$$

$$\Updownarrow$$

Discrete energy equality + alignment

$$\phi(u_i) + \tau_i \left\| \frac{u_i - u_{i-1}}{\tau_i} \right\|^2 = \phi(u_{i-1})$$

$$(u_i - u_{i-1}) \quad \text{parallel to} \quad \mathrm{D}\phi(u_i)$$

1: the Romero scheme

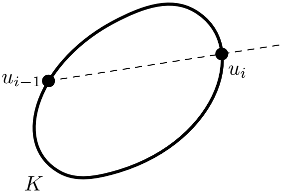
Discrete energy equality + alignment

$$\phi(u_i) + \tau_i \left\| \frac{u_i - u_{i-1}}{\tau_i} \right\|^2 = \phi(u_{i-1})$$

$$(u_i - u_{i-1}) \quad \text{parallel to} \quad \mathrm{D}\phi(u_i)$$

Existence for $\phi \in C^1(\mathbb{R}^n)$ bounded below:

1. $K = \{\phi(u) + \tau_i \|(u - u_{i-1})/\tau_i\|^2 = \phi(u_{i-1})\}$ is compact
2. Let u_i maximize $u \mapsto \|u - u_{i-1}\|^2$ on K
3. If $\mathrm{D}\phi(u_{i-1}) \neq 0$, K does not reduce to u_{i-1}. Hence, $u_i \neq u_{i-1}$



1: the Romero scheme

The Romero scheme

$$\frac{u_i - u_{i-1}}{\tau_i} + \mathrm{D}\phi(u_i) + \Big(\phi(u_i) - \phi(u_{i-1}) - (\mathrm{D}\phi(u_i), u_i - u_{i-1})\Big)\frac{u_i - u_{i-1}}{\|u_i - u_{i-1}\|^2} = 0$$

Convergence for $\phi \in C^2(\mathbb{R}^n)$:

$$\left\|\Big(\phi(u_i) - \phi(u_{i-1}) - (\mathrm{D}\phi(u_i), u_i - u_{i-1})\Big)\frac{u_i - u_{i-1}}{\|u_i - u_{i-1}\|^2}\right\|$$

$$\leqslant \frac{1}{2}\|\mathrm{D}^2\phi\|\,\|u_i - u_{i-1}\| = \frac{\tau_i}{2}\underbrace{\|\mathrm{D}^2\phi\|}_{L^\infty}\underbrace{\left\|\frac{u_i - u_{i-1}}{\tau_i}\right\|}_{L^2}$$

1: the Romero scheme

Shortcomings:

- Analysis for smooth functions with compact sublevels
- Essentially restricted to finite dimensions
- Two equations: balance + alingment
- Conditions on τ in the nonsmooth case
- Nonvariational

Euler scheme

$$\frac{u_i - u_{i-1}}{\tau_i} + \mathrm{D}\phi(u_i) = 0$$

Discrete energy equality

$$\phi(u_i) + \tau_i \left\| \frac{u_i - u_{i-1}}{\tau_i} \right\|^2 = \phi(u_{i-1})$$

$$\Updownarrow$$

Discrete energy equality 2

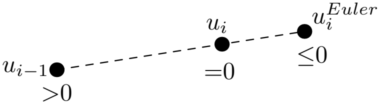
$$\phi(u_i) + \frac{\tau_i}{2} \left\| \frac{u_i - u_{i-1}}{\tau_i} \right\|^2 + \frac{\tau_i}{2} \|\mathrm{D}\phi(u_i)\|^2 = \phi(u_{i-1})$$

2: the De Giorgi scheme

The De Giorgi scheme

$$\phi(u_i) + \frac{\tau_i}{2}\left\|\frac{u_i - u_{i-1}}{\tau_i}\right\|^2 + \frac{\tau_i}{2}\|\mathrm{D}\phi(u_i)\|^2 = \phi(u_{i-1}) + \alpha_i$$

- Existence for ϕ convex ($\alpha_i = 0$) or ϕ smooth ($\alpha_i = \tau^2$)



- Convergence if $\sum_i \alpha_i^+ \to 0$

2: the De Giorgi scheme

$$
\begin{aligned}
0 &= \lim \sum_i \alpha_i^+ = \\
&\geqslant \liminf \left(\phi(u_m) + \frac{1}{2}\sum_{i=1}^{m} \tau_i \left\| \frac{u_i - u_{i-1}}{\tau_i} \right\|^2 + \frac{1}{2}\sum_{i=1}^{m} \tau_i \|\mathrm{D}\phi(u_i)\|^2 - \phi(u_0) \right) \\
&\geqslant \phi(u(t)) + \frac{1}{2}\int_0^t \|u'\|^2 + \frac{1}{2}\int_0^t \|\mathrm{D}\phi(u)\|^2 - \phi(u_0) \\
&= \int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}t}\phi(u) + \frac{1}{2}\|u'\|^2 + \frac{1}{2}\|\mathrm{D}\phi(u)\|^2 \right) \\
&= \int_0^t \left((\mathrm{D}\phi(u), u') + \frac{1}{2}\|u'\|^2 + \frac{1}{2}\|\mathrm{D}\phi(u)\|^2 \right) \\
&= \frac{1}{2}\int_0^t \|u' + \mathrm{D}\phi(u)\|^2 \qquad\qquad \Rightarrow u' + \mathrm{D}\phi(u) = 0
\end{aligned}
$$

2: the De Giorgi scheme

- gradient flows $u' + \mathrm{D}\phi(u) = 0$
- doubly nonlinear flows $\mathrm{D}\psi(u') + \mathrm{D}\phi(u) = 0$
- generalized gradient flows $\mathrm{D}\psi(u, u') + \mathrm{D}\phi(u) = 0$
- GENERIC flows $\mathrm{D}\psi(u' + J\,\mathrm{D}E(u)) + \mathrm{D}\phi(u) = 0$

  General Equations for Non-Equilibrium Reversible-Irreversible Coupling

  [Grmela, Öttinger, 97]

The De Giorgi scheme

$$\phi(u_i) + \tau_i \psi\left(\frac{u_i - u_{i-1}}{\tau_i} + J\,\mathrm{D}E(u_i)\right) + \tau_i \psi^*(-\mathrm{D}\phi(u_i)) = \phi(u_{i-1})$$

Conclusions

- Exact dissipation dynamics can be replicated at the discrete level by the Romero and the De Giorgi scheme
- The Romero scheme is well suited for smooth, finite-dimensional energies
- The De Giorgi scheme works for smooth perturbations of convex energies in infinite dimension and can be extended to more general evolutions

http://www.mat.univie.ac.at/~stefanelli